Kuwait University — Math 102 (Calculus II) — August 7, 2010

Department of Mathematics — Final Examination — Duration: 120 minutes

**Calculators, telephones, and pagers are not allowed.**

Answer all of the following questions. Read each question carefully. Justify all your answers.

1. Let

$$f(x) = \begin{cases} \ln(x+1) & \text{if } -1 < x < 0 \\ \exp(-x^2) & \text{if } x \geq 0. \end{cases}$$

Show that $f$ is one-to-one on $(-1, \infty)$. **(2 points)**

2. Evaluate $\displaystyle\lim_{x\to\infty} \frac{\pi - 2\tan^{-1} x}{\pi - 2\sec^{-1} x}$. **(3 points)**

3. Evaluate the following integrals. **(3 points each)**

(a) $\displaystyle\int x^3 \arctan x \, dx$

(b) $\displaystyle\int \frac{1}{3\cosh x + 4\sinh x + 3} dx$

(c) $\displaystyle\int \frac{1 - \sin x}{(x^2 + 2x\cos x - \sin^2 x)^{3/2}} dx$

4. Determine whether the improper integral $\displaystyle\int_0^{\pi} \left( \frac{1 + 2\cos x}{x + 2\sin x} - \frac{1}{x} \right) dx$ is convergent or divergent, and find its value if it is convergent. **(3 points)**

5. Find $b$ so that $\displaystyle\int \frac{bx+1}{x^2(x+1)^3} dx$ is a rational function, and evaluate the integral for every such value of $b$. **(2 points)**

6. Find the length of the curve $y = \ln\sqrt{\tanh x}$, $1 \leq x \leq 2$. **(4 points)**

7. Find the centroid of the region bounded by the curves $y = 1/(x^2 - 2x + 5)$, $y = 0$, $x = -1$, and $x = 3$. **(4 points)**

8. Let $C$ be the curve given by

$$x = \tfrac{2}{3}(1-t)^{3/2}, \quad y = 2(1+t)^{1/2}, \qquad 0 \leq t \leq 1.$$

(a) Find $d^2y/dx^2$. Is $C$ concave upward or downward? **(3 points)**

(b) Find the area of the surface obtained by rotating $C$ about the $x$-axis. **(3 points)**

9. (a) Show that the polar equation $r = a\sin\theta + b\cos\theta$ represents a circle whenever $a$ and $b$ are not both zero, and find its center and radius. **(2 points)**

(b) Find the area of the region that lies in the second quadrant, inside the graph of the polar equation $r = 1 + \cos\theta$, and outside the graph of the polar equation $r = \sin\theta + \cos\theta$. **(5 points)**

Kuwait University Department of Mathematics

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# SOLUTION

1. Let $x_1 \in (-1, \infty)$, $x_2 \in (-1, \infty)$, and $x_2 \neq x_1$.
   If $x_1 < 0$ and $x_2 \geq 0$, then $f(x_1) < 0$ and $f(x_2) > 0$.
   If $x_1 < 0$ and $x_2 < 0$, then $\ln(x_1 + 1) \neq \ln(x_2 + 1)$.
   If $x_1 \geq 0$ and $x_2 \geq 0$, then $\exp(-x_1^2) \neq \exp(-x_2^2)$.
   So, in all possible cases, if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$.

2. The limit has the indeterminate form 0/0. For $x > 1$,

$$\frac{\frac{d}{dx}(\pi - 2\tan^{-1} x)}{\frac{d}{dx}(\pi - 2\sec^{-1} x)} = \frac{-\dfrac{2}{1+x^2}}{-\dfrac{2}{x\sqrt{x^2-1}}} = \frac{x\sqrt{x^2-1}}{1+x^2} = \frac{\sqrt{1-x^{-2}}}{x^{-2}+1} \to 1 \quad \text{as } x \to \infty.$$

   Thus, l'Hospital's Rule is applicable, and gives $\displaystyle\lim_{x\to\infty} \frac{\pi - 2\tan^{-1} x}{\pi - 2\sec^{-1} x} = 1$.

3. (a) Integrate by parts with $u = \arctan x$ and $dv = x^3\,dx$. Then $du = [1/(1+x^2)]\,dx$ and $v = x^4/4$. So that

$$\begin{aligned}
\int x^3 \arctan x\,dx &= \frac{x^4}{4}\arctan x - \frac{1}{4}\int \frac{x^4}{1+x^2}\,dx \\
&= \frac{x^4}{4}\arctan x - \frac{1}{4}\int \left(x^2 - 1 + \frac{1}{x^2+1}\right)\,dx \\
&= \frac{x^4}{4}\arctan x - \frac{1}{4}\left(\frac{x^3}{3} - x + \arctan x\right) + C \\
&= -\tfrac{1}{12}x^3 + \tfrac{1}{4}x + \tfrac{1}{4}(x^4 - 1)\arctan x + C.
\end{aligned}$$

   (b) By the definition of the hyperbolic functions,

$$\begin{aligned}
\int \frac{1}{3\cosh x + 4\sinh x + 3}\,dx &= \int \frac{1}{3(e^x + e^{-x})/2 + 4(e^x - e^{-x})/2 + 3}\,dx \\
&= \cdots = \int \frac{2e^x}{7e^{2x} + 6e^x - 1}\,dx.
\end{aligned}$$

   Substitute $u = e^x$ which implies that $du = e^x\,dx$. Then

$$\begin{aligned}
\int \frac{1}{3\cosh x + 4\sinh x + 3}\,dx &= \int \frac{2}{7u^2 + 6u - 1}\,du = \int \frac{2}{(7u-1)(u+1)}\,du \\
&= \frac{1}{4}\int \left(\frac{7}{7u-1} - \frac{1}{u+1}\right)\,du \\
&= \tfrac{1}{4}\left(\ln|7u-1| - \ln|u+1|\right) + C \\
&= \frac{1}{4}\ln\left|\frac{7e^x - 1}{e^x + 1}\right| + C.
\end{aligned}$$

(c) By completing the square,

$$\begin{aligned}\int \frac{1-\sin x}{\left(x^2+2x\cos x-\sin^2 x\right)^{3/2}}\,dx &= \int \frac{1-\sin x}{\left[(x+\cos x)^2-\cos^2 x-\sin^2 x\right]^{3/2}}\,dx \\ &= \int \frac{1-\sin x}{\left[(x+\cos x)^2-1\right]^{3/2}}\,dx.\end{aligned}$$

Substitute $x+\cos x=\sec\theta$ which implies that $\sqrt{(x+\cos x)^2-1}=\tan\theta$ and $(1-\sin x)\,dx=\sec\theta\tan\theta\,d\theta$. This yields

$$\begin{aligned}\int \frac{1-\sin x}{\left(x^2+2x\cos x-\sin^2 x\right)^{3/2}}\,dx &= \int \frac{\sec\theta\tan\theta}{\tan^3\theta}\,d\theta=\int\frac{\cos\theta}{\sin^2\theta}\,d\theta \\ &= -\frac{1}{\sin\theta}+C=-\frac{\sec\theta}{\tan\theta}+C \\ &= -\frac{x+\cos x}{\sqrt{(x+\cos x)^2-1}}+C.\end{aligned}$$

4. The integrand is continuous on $(0,\pi]$ but not on $[0,\pi]$. So, consider

$$\int_t^\pi\left(\frac{1+2\cos x}{x+2\sin x}-\frac{1}{x}\right)dx=\left[\ln(x+2\sin x)-\ln x\right]\Big|_t^\pi=-\ln\left(\frac{t+2\sin t}{t}\right)$$

for $t\in(0,\pi)$. This gives

$$\lim_{t\to0+}\int_t^\pi\left(\frac{1+2\cos x}{x+2\sin x}-\frac{1}{x}\right)dx=-\lim_{t\to0+}\ln\left(1+2\frac{\sin t}{t}\right)=-\ln 3.$$

Answer: The improper integral is convergent and its value is $-\ln 3$.

5. The partial fraction decomposition of the integrand for arbitrary $b$ is

$$\frac{bx+1}{x^2(x+1)^3}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}+\frac{D}{(x+1)^2}+\frac{E}{(x+1)^3}.$$

For the integral to be a rational function, necessarily $A=C=0$. So,

$$\frac{bx+1}{x^2(x+1)^3}=\frac{B}{x^2}+\frac{D}{(x+1)^2}+\frac{E}{(x+1)^3},$$

where

$$\begin{aligned}bx+1 &= B(x+1)^3+Dx^2(x+1)+Ex^2 \\ &= (B+D)x^3+(3B+D+E)x^2+3Bx+B.\end{aligned}$$

Equating coefficients,

$$\left\{\begin{aligned}B+D&=0\\3B+D+E&=0\\3B&=b\\B&=1\end{aligned}\right.\quad\Longrightarrow\quad\left\{\begin{aligned}B&=1\\D&=-1\\E&=-2\\b&=3.\end{aligned}\right.$$

So $b = 3$ and

$$\begin{aligned}\int \frac{3x+1}{x^2(x+1)^3}\,dx &= \int \left(\frac{1}{x^2} - \frac{1}{(x+1)^2} - \frac{2}{(x+1)^3}\right) dx \\ &= -\frac{1}{x} + \frac{1}{x+1} + \frac{1}{(x+1)^2} + K = \frac{-1}{x(x+1)^2} + K.\end{aligned}$$

6. Since $y = \frac{1}{2}\ln\tanh x$,

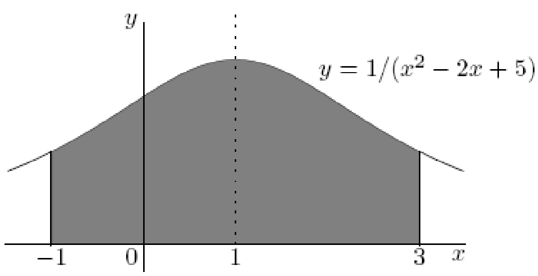
$$\frac{dy}{dx} = \frac{1}{2}\frac{\operatorname{sech}^2 x}{\tanh x} = \frac{1}{2\cosh x\sinh x} = \frac{1}{\sinh 2x},$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{1}{\sinh^2 2x} = \frac{\sinh^2 2x + 1}{\sinh^2 2x} = \frac{\cosh^2 2x}{\sinh^2 2x},$$

and the length of the curve is

$$\begin{aligned}\int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx &= \int_1^2 \frac{\cosh 2x}{\sinh 2x}\,dx = \frac{1}{2}\ln\sinh 2x\Big|_1^2 = \frac{1}{2}\ln\left(\frac{\sinh 4}{\sinh 2}\right) \\ &= \ln\sqrt{2\cosh 2}.\end{aligned}$$

7. Completing the square, $x^2 - 2x + 5 = (x-1)^2 + 4$. So, $1/(x^2 - 2x + 5) > 0$ for all $x$.




Let $A$ be the area of the region, and $(\bar{x}, \bar{y})$ be the coordinates of its centroid.
Because the region is symmetric with respect to the line $x = 1$, there holds $\bar{x} = 1$.

$$A = \int_{-1}^{3} \frac{1}{(x-1)^2 + 4}\,dx = 2\int_1^3 \frac{1}{(x-1)^2 + 4}\,dx$$

and

$$\bar{y} = \frac{1}{2A}\int_{-1}^{3} \frac{1}{[(x-1)^2 + 4]^2}\,dx = \frac{1}{A}\int_1^3 \frac{1}{[(x-1)^2 + 4]^2}\,dx.$$

Substitute $x - 1 = 2\tan\theta$ which implies that $(x-1)^2 + 4 = 4\sec^2\theta$ and $dx = 2\sec^2\theta\,d\theta$.
So that

$$A = 2\int_0^{\pi/4} \frac{1}{4\sec^2\theta} 2\sec^2\theta\,d\theta = \int_0^{\pi/4} 1\,d\theta = \frac{\pi}{4}$$

and

$$\begin{aligned}\bar{y} &= \frac{4}{\pi}\int_0^{\pi/4}\frac{1}{(4\sec^2\theta)^2}2\sec^2\theta\,d\theta = \frac{1}{2\pi}\int_0^{\pi/4}\cos^2\theta\,d\theta \\ &= \frac{1}{4\pi}\int_0^{\pi/4}(1+\cos 2\theta)\,d\theta = \frac{1}{8\pi}(2\theta+\sin 2\theta)\Big|_0^{\pi/4} = \cdots = \frac{\pi+2}{16\pi}.\end{aligned}$$

Answer: The centroid of the region is $(1, \frac{\pi+2}{16\pi})$.

8. Differentiating,

$$\frac{dx}{dt} = -(1-t)^{1/2} \quad \text{and} \quad \frac{dy}{dt} = (1+t)^{-1/2}.$$

(a) This gives

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{(1+t)^{-1/2}}{-(1-t)^{1/2}} = -(1-t^2)^{-1/2},$$

and,

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(-(1-t^2)^{-1/2}\right)}{-(1-t)^{1/2}} = \frac{-t(1-t^2)^{-3/2}}{-(1-t)^{1/2}} \\ &= \frac{t}{(1-t)^2(1+t)^{3/2}}.\end{aligned}$$

It follows that $d^2y/dx^2 > 0$ for $0 < t < 1$. So $C$ is concave upward.

(b) Let $s$ denote the arc length. Then the surface area is

$$\begin{aligned}\int_C 2\pi y\,ds &= 2\pi\int_0^1 y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt \\ &= 2\pi\int_0^1 2(1+t)^{1/2}\sqrt{1-t+(1+t)^{-1}}\,dt = \cdots = 4\pi\int_0^1\sqrt{2-t^2}\,dt.\end{aligned}$$

Substitute $t = \sqrt{2}\sin\theta$ which implies that $\sqrt{2-t^2} = \sqrt{2}\cos\theta$ and $dt = \sqrt{2}\cos\theta\,d\theta$. So that

$$\begin{aligned}\int_C 2\pi y\,ds &= 4\pi\int_0^{\pi/4}(\sqrt{2}\cos\theta)^2\,d\theta = 4\pi\int_0^{\pi/4}(1+\cos 2\theta)\,d\theta \\ &= 2\pi(2\theta+\sin 2\theta)\Big|_0^{\pi/4} = 2\pi\left(\tfrac{\pi}{2}+1\right) = \pi(\pi+2).\end{aligned}$$

9. (a) $r = a\sin\theta + b\cos\theta \implies r^2 = ar\sin\theta + br\cos\theta \implies x^2+y^2 = ay+bx \implies$

$$(x-b/2)^2 + (y-a/2)^2 = (a^2+b^2)/4.$$

This is the equation of a circle with center $(b/2, a/2)$ and radius $\sqrt{a^2+b^2}/2$.

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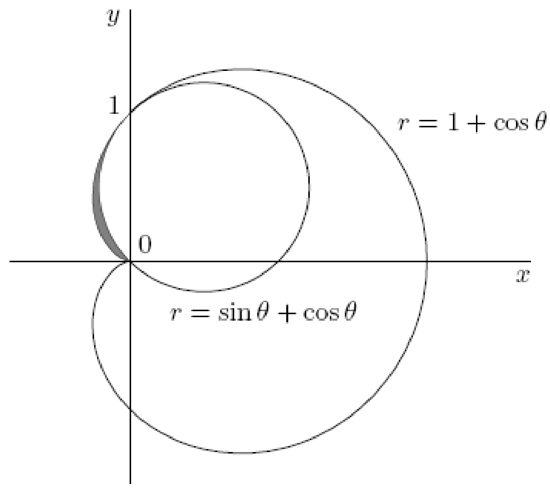
(b) The polar equation $r = \sin\theta + \cos\theta$ represents the circle with center $(\frac{1}{2}, \frac{1}{2})$ and radius $\frac{1}{\sqrt{2}}$. The polar equation $r = 1 + \cos\theta$ describes a cardioid.




The graphs of the two equations do not intersect inside the second quadrant since $1 + \cos\theta > \sin\theta + \cos\theta$ for $\pi/2 < \theta < \pi$ and for $3\pi/2 < \theta < 2\pi$.
Let $A_1$ be the area in the second quadrant inside the graph of the cardioid, and $A_2$ be the area in the second quadrant inside the circle.
Then

$$\begin{aligned} A_1 &= \frac{1}{2}\int_{\pi/2}^{\pi}(1+\cos\theta)^2\,d\theta = \tfrac{1}{2}\int_{\pi/2}^{\pi}(1+2\cos\theta+\cos^2\theta)\,d\theta \\ &= \tfrac{1}{4}\int_{\pi/2}^{\pi}(3+4\cos\theta+\cos 2\theta)\,d\theta = \tfrac{1}{8}(6\theta+8\sin\theta+\sin 2\theta)\Big|_{\pi/2}^{\pi} = \frac{3\pi-8}{8}. \end{aligned}$$

On the other hand,[1] because $A_2$ is one quarter of the area inside a circle of radius $\frac{1}{\sqrt{2}}$ and outside an inscribed square of side length 1,

$$A_2 = \frac{1}{4}\left[\pi\left(\tfrac{1}{\sqrt{2}}\right)^2 - 1^2\right] = \frac{\pi-2}{8}.$$

Answer: The area is $A_1 - A_2 = \cdots = (\pi - 3)/4$.

[1]Alternatively,

$$A_2 = \frac{1}{2}\int_{\pi/2}^{3\pi/4}(\sin\theta+\cos\theta)^2\,d\theta = \frac{1}{2}\int_{\pi/2}^{3\pi/4}(1+\sin 2\theta)\,d\theta = \frac{1}{4}(2\theta-\cos 2\theta)\Big|_{\pi/2}^{3\pi/4} = \cdots = \frac{\pi-2}{8}.$$